3D compatible ternary systems and Yang-Baxter maps

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Abstract

According to Shibukawa, ternary systems defined on quasigroups and satisfying certain conditions provide a way of constructing dynamical Yang-Baxter maps. After noticing that these conditions can be interpreted as 3-dimensional compatibility of equations on quad-graphs, we investigate when the associated dynamical Yang-Baxter maps are in fact parametric Yang-Baxter maps. In some cases these maps can be obtained as reductions of higher dimensional maps through compatible constraints. Conversely, parametric YB maps on quasigroups with an invariance condition give rise to 3-dimensional compatible systems. The application of this method on spaces with certain quasigroup structures provides new examples of multi-parametric YB maps and 3-dimensional compatible systems.

1 Introduction

In [16, 17], Shibukawa studied the (set–theoretic) dynamical Yang-Baxter (YB) equation on quasigroups and on loops. A construction of dynamical YB maps was introduced in [17], from ternary systems defined on left quasigroups that satisfy certain compatibility conditions. A parametric version of the latter conditions turns to be equivalent with the 3-dimensional consistency property of discrete (parametric) equations on lattices. Extra 'symmetry' conditions on the ternary system provide a way of constructing parametric YB maps on groups and on particular quasigroups. From the other hand, parametric YB maps satisfying an invariance condition on quasigroups give rise to 3-dimensional compatible systems. We apply these constructions on spaces with certain quasigroup structures in order to derive (multi-)parametric YB maps from 3-dimensional consistent equations on quad-graphs and vice versa.

We begin in Section 2 by giving the necessary definitions for dynamical YB maps, parametric YB maps and Lax matrices and by presenting the construction of dynamical YB maps due to Shibukawa.

In Section 3, we show the equivalence between the 3-dimensonal consistency property and the conditions of the previous construction. By considering symmetry conditions on 3-dimensional consistent equations on quasigroups we derive parametric YB maps. We apply this method on 3-dimensional consistent equations on quad-graphs. In each case, the corresponding quasigroup structure that we consider is chosen according to the fulfilled symmetry condition and gives rise to a different YB map.

In section 4 we study the inverse problem in order to derive 3-dimensional compatible systems from YB maps. We apply this construction on the quasigroup $\mathbb{C}^*$ equipped with

division to a new four parametric quadrirational YB map on $\mathbb{C}^* \times \mathbb{C}^*$. The latter map was obtained from a higher dimensional map by imposing a compatible constraint. The corresponding 3-dimensional compatible system provides another four parametric map on the loop $\mathbb{C}$ equipped with substraction. We close this section by presenting a 3-dimensional compatible system on $GL_2(\mathbb{C})$ (that can be generalized on $GL_n(\mathbb{C})$), derived by the general solution of the re-factorization problem presented in [5, 6].

We conclude in Section 5 by giving some comments and perspectives for future work.

2 Quasigroups, Ternary systems and YB maps

Let H, X be two non-empty sets and ϕ a map from $H \times X$ to H. A map

$$R(\lambda) : X \times X \to X \times X, \quad \lambda \in H,$$

is called *dynamical Yang-Baxter map*, with respect to H, X and ϕ, if for any $\lambda \in H$ the map $R(\lambda)$ satisfies the *dynamical YB equation*

$$R_{23}(\lambda)R_{13}(\phi(\lambda, X^{(2)}))R_{12}(\lambda) = R_{12}(\phi(\lambda, X^{(3)}))R_{13}(\lambda)R_{23}(\phi(\lambda, X^{(1)})). \tag{1}$$

The maps $R_{12}(\lambda)$, $R_{12}(\phi(\lambda, X^{(3)}))$ etc, are defined on the triple Cartesian product $X \times X \times X$, as

$$R_{12}(\lambda)(u, v, w) = (R(\lambda)(u, v), w), \;\; R_{12}(\phi(\lambda, X^{(3)}))(u, v, w) = (R(\phi(\lambda, w))(u, v), w),$$

for $u, v, w \in X$, and in a similar way R_{13} and R_{23}. The parameter λ is called the dynamical parameter of the map $R(\lambda)$ (and must not be confused with the parameters α, β of parametric YB maps). If $R(\lambda)$ does not depend on λ, then it is a YB map, that is a map $R : X \times X \to X \times X$ that satisfies the YB equation, $R_{23}R_{13}R_{12} = R_{12}R_{13}R_{23}$, where by R_{ij} for $i, j = 1, ..., 3$, we denote the action of the map R on the i and j factor of $X \times X \times X$.

A YB map $R : (X \times I) \times (X \times I) \to (X \times I) \times (X \times I)$, with

$$R : ((x, \alpha), (y, \beta)) \mapsto ((u, \alpha), (v, \beta)) = ((u(x, \alpha, y, \beta), \alpha), (v(x, \alpha, y, \beta), \beta)), \tag{2}$$

is called a *parametric YB map* ([18, 19, 20]). We usually keep the parameters separately and denote a parametric YB map as $R_{\alpha,\beta}(x, y) : X \times X \to X \times X$. From our point of view, the sets X and I have the structure of an algebraic variety and the considered maps are birational. A *Lax Matrix* of the YB map (2) is a map $L : X \times I \to Mat(n \times n)$, that depends on a spectral parameter $\zeta \in \mathbb{C}$, such that

$$L(u; \alpha)L(v; \beta) = L(y; \beta)L(x; \alpha). \tag{3}$$

Furthermore, if equation (3) is equivalent to $(u,\ v) = R_{\alpha,\beta}(x, y)$ then $L(x; \alpha)$ will be called *strong Lax matrix.*

Definition 2.1. *A non empty set L with a binary operation $\cdot : L \times L \to L$ is called left quasigroup $(L, \cdot)$, if for every $u, w \in L$ there is a unique $v \in L$ such that $u \cdot v = w$ (right quasigroups are defined in an analogous way).*

From this definition it turns out that in every left quasigroup one additional operation is defined, namely $\setminus : L \times L \to L$, called *left division*, with $u \setminus w = v$ if and only if $u \cdot v = w$.

A left quasigroup $(L, \cdot)$ with the property stating that for every $v, w \in L$ there is a unique $u \in L$ such that $u \cdot v = w$, is called *quasigroup*. In other words a quasigroup is a left and right quasigroup. If there exist an element e_l of a quasigroup L such that $e_l \cdot u = u$ for every $u \in L$, then it is called *left identity*, similarly $e_r \in L$ is called right identity if $u \cdot e_r = u$, for every $u \in L$. Furthermore, if there is an element $e \in L$ such that $u \cdot e = e \cdot u = u$, for any $u \in G$, i.e. the left and right identities coincide, then $(L, \cdot)$ is called *loop*. We usually omit the symbol $\cdot$ and write just uv for $u \cdot v$.

Quasigroups and loops can be regarded as generalizations of groups. In particular an associative loop is a group. For a detailed study on quasigroups and loops we refer to [15].

Next, we present the construction of dynamical YB maps, introduced by Shibukawa in [17]. We begin with the following definition.

Definition 2.2. *A non-empty set M equipped with a ternary operation $\mu : M \times M \times M \to M$ is called a* ternary system (M, μ).

Let $(L, \cdot)$ be a left quasigroup, (M, μ) a ternary system and a bijective map $\pi : L \to M$. We consider the map $R_\lambda : L \times L \to L \times L$, with

$$R_\lambda(x, y) = (\eta_\lambda(y)(x), \xi_\lambda(x)(y)), \tag{4}$$

where

$$\xi_\lambda(x)(y) = \lambda \setminus \pi^{-1}(\mu(\pi(\lambda), \pi(\lambda x), \pi((\lambda x)y))),$$
$$\eta_\lambda(x)(y) = (\lambda \xi_\lambda(y)(x)) \setminus ((\lambda y)x), \quad \lambda,\ x,\ y \in L.$$

Theorem 2.1. (*Shibukawa*) *The map R_λ (4) is a dynamical YB map with respect to L, L and $\phi : L \times L \to L$, with $\phi(\lambda, x) = \lambda x$, if and only if*

$$\begin{aligned} \mu(a, \mu(a, b, c), \mu(\mu(a, b, c), c, d)) &= \mu(a, b, \mu(b, c, d)) \\ \mu(\mu(a, b, \mu(b, c, d)), \mu(b, c, d), d) &= \mu(\mu(a, b, c), c, d)) \end{aligned} \tag{5}$$

for every $a, b, c, d \in M$.

The dynamical YB map (4) is a generalization of the YB map that was presented in [8], on any group that acts on itself and the action satisfies a compatibility condition.

3 3–Dimensional Consistency and YB Maps

We are going to show that the conditions of theorem 2.1 can be interpreted as 3-dimensional consistency property of a quad-graph equation.

We consider a parametric ternary operation $\mu_{\alpha,\beta} : X \times X \times X \to X$, $\alpha, \beta \in I$, the corresponding parametric equation

$$w = \mu_{\alpha,\beta}(a, b, c), \tag{6}$$

the initial values a, b, c, d placed on the vertices of a cube and the parameters α, β, γ assigned to the edges, as shown in Fig.1. All parallel edges carry the same parameter.

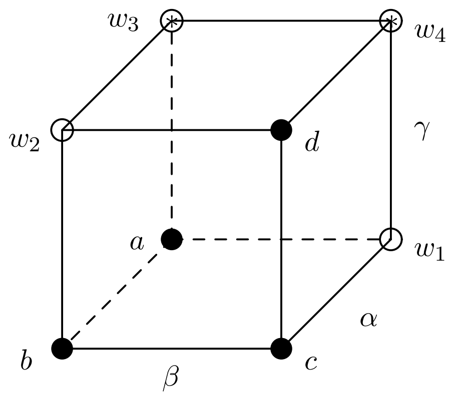

Figure 1: Consistency around the cube

The values w_1 and w_2 are determined uniquely from the initial conditions on the cube and equation (6) as follows

$$w_1 = \mu_{\alpha,\beta}(a,b,c), \ \ w_2 = \mu_{\beta,\gamma}(b,c,d).$$

From the other hand, the values w_3 and w_4 can be derived by two different ways. Following Fig.1 we have that

$$\begin{aligned} w_3 &= \mu_{\beta,\gamma}(a, w_1, \mu_{\alpha,\gamma}(w_1, c, d)) \text{ or } w_3 = \mu_{\alpha,\gamma}(a, b, w_2), \\ w_4 &= \mu_{\alpha,\beta}(\mu_{\alpha,\gamma}(a, b, w_2), w_2, d) \text{ or } w_4 = \mu_{\alpha,\gamma}(w_1, c, d)). \end{aligned}$$

We will say that the equation (6) satisfies the *3-dimensional (3D) consistency or compatibility condition* if w_3 and w_4 are uniquely defined, that is the ternary operation satisfies

$$\mu_{\beta,\gamma}(a, \mu_{\alpha,\beta}(a,b,c), \mu_{\alpha,\gamma}(\mu_{\alpha,\beta}(a,b,c),c,d)) = \mu_{\alpha,\gamma}(a,b,\mu_{\beta,\gamma}(b,c,d)), \quad (7)$$

$$\mu_{\alpha,\beta}(\mu_{\alpha,\gamma}(a,b,\mu_{\beta,\gamma}(b,c,d)), \mu_{\beta,\gamma}(b,c,d),d) = \mu_{\alpha,\gamma}(\mu_{\alpha,\beta}(a,b,c),c,d)). \quad (8)$$

This form of 3D compatibility can be traced back in [3] (see also [13]). An equivalent formulation was introduced both in [9] (in order to derive Lax pair for the discrete Krichever–Novikov equation) and in [4], giving rise to the ABS classification [1] of integrable D_4-symmetric quad-graph equations.

Equations (7) and (8) are a parametric version of the conditions of theorem 2.1. According to this observation we give the next definition.

Definition 3.1. *A non-empty set X equipped with a parametric ternary operation $\mu_{\alpha,\beta}$: $X \times X \times X \to X$ that satisfies (7) and (8), is called 3D compatible ternary system.*

So, according to theorem 2.1, every 3D compatible ternary system on a quasigroup gives rise to a dynamical YB map. A natural question is to investigate instances where the dynamical YB maps obtained in this way turn out to be independent of the dynamical parametre λ i.e. they are YB maps. In order to eliminate the dynamical parameter λ some extra symmetry conditions on equations are helpful. In case of groups because of associativity, it is enough to consider homogeneous 3D consistent equations in the following sense.

Definition 3.2. *A 3D compatible ternary system on a quasigroup* $(L,\cdot)$ *is called homogeneous if* $\mu_{\alpha,\beta}(\lambda a,\lambda b,\lambda c)=\lambda\mu_{\alpha,\beta}(a,b,c)$, *for every* $a,b,c,\lambda\in L$ *and* $\alpha,\beta\in I$.

The next proposition gives the parametric version of theorem 2.1, for homogeneous 3D compatible ternary systems on groups, that yield parametric YB maps.

Proposition 3.3. *Let* $(L,\mu_{\alpha,\beta})$ *be a homogeneous 3D compatible ternary system on the group* L. *The map*

$$R_{\alpha,\beta}(x,y)=((\mu_{\alpha,\beta}(e,x,xy))^{-1}xy,\mu_{\alpha,\beta}(e,x,xy)) \tag{9}$$

is a parametric YB map.

The proof of this proposition is given below together with the proof of proposition 3.4. The corresponding YB map for Abelian groups with additive notation becomes

$$R_{\alpha,\beta}(x,y)=((x+y-\mu_{\alpha,\beta}(0,x,x+y)),\mu_{\alpha,\beta}(0,x,x+y)). \tag{10}$$

In the more general case of quasigroups the homogeneous condition is not always sufficient. We are not going to investigate under which algebraic conditions one can eliminate the dynamical parameter, but instead we consider the following particular cases.

We consider the set $L=\mathbb{C}\setminus\{0\}$ with the binary operation $a*b=\frac{b}{a}$. This is a quasigroup with left identity element $e_l=1$. In this quasigroup a symmetry condition of 3D compatible ternary systems that provides parametric YB maps is given below.

Proposition 3.4. *Let* $(L,\mu_{\alpha,\beta})$ *be a 3D compatible ternary system on the quasigroup* $(L,*)$, *such that*

$$\lambda\mu_{\alpha,\beta}(a,b,c)=\mu_{\alpha,\beta}(\frac{a}{\lambda},\lambda b,\frac{c}{\lambda}),\quad \text{for every } \lambda\in L \text{ and } \alpha,\beta\in I, \tag{11}$$

then the map

$$R_{\alpha,\beta}(x,y)=(\frac{y}{x}\mu_{\alpha,\beta}(1,x,\frac{y}{x}),\mu_{\alpha,\beta}(1,x,\frac{y}{x})) \tag{12}$$

is a parametric YB map.

Proof. We present the proof of propositions 3.3 and 3.4. Let us denote by $x',x'',y',y'',z',z'',\tilde{x},...,\tilde{\tilde{z}}$ the values defined each time by the corresponding map (9) or (12) as follows,

$$R^{12}_{\alpha\beta}(x,y,z)=(x',y',z),R^{13}_{\alpha,\gamma}R^{12}_{\alpha\beta}(x,y,z)=(x'',y',z'),R^{23}_{\beta\gamma}R^{13}_{\alpha,\gamma}R^{12}_{\alpha\beta}(x,y,z)=(x'',y'',z''),$$
$$R^{23}_{\beta\gamma}(x,y,z)=(x,\tilde{y},\tilde{z}),\quad R^{13}_{\alpha\gamma}R^{23}_{\beta\gamma}(x,y,z)=(\tilde{x},\tilde{y},\tilde{\tilde{z}}),\quad R^{12}_{\alpha\beta}R^{13}_{\alpha,\gamma}R^{23}_{\beta\gamma}(x,y,z)=(\tilde{\tilde{x}},\tilde{\tilde{y}},\tilde{\tilde{z}}).$$

For convenience, we consider the map $T_{\alpha,\beta}:L\times L\times L\to L\times L$, defined by

$$T_{\alpha,\beta}:(a,b,c)\mapsto(\mu_{\alpha,\beta}(a,b,c)\setminus c,a\setminus\mu_{\alpha,\beta}(a,b,c)). \tag{13}$$

In Prop. 3.3, where L is a group $a\setminus b=a^{-1}b$, while for the quasigroup of Prop. 3.4 $a\setminus b=ab$, for every $a,b\in L$. In both cases we can verify that $R_{\alpha,\beta}(x,y)=T_{\alpha,\beta}(e_l,x,x*y)$, where e_l is the corresponding left identity element (the identity element e for Prop. 3.3 and 1 for Prop. 3.4) and by $*$ we denote the corresponding binary quasigroup operation in both cases. Next, we set

$$\begin{aligned}
&w_1:=\mu_{\alpha,\beta}(e_l,x,x*y),\ \ w_2:=\mu_{\beta,\gamma}(x,x*y,(x*y)*z),\\
&w_3:=\mu_{\beta,\gamma}(e_l,w_1,\mu_{\alpha,\gamma}(w_1,x*y,(x*y)*z))=\mu_{\alpha,\gamma}(e_l,x,w_2),\\
&w_4:=\mu_{\alpha,\beta}(\mu_{\alpha,\gamma}(e_l,x,w_2),w_2,(x*y)*z)=\mu_{\alpha,\gamma}(w_1,x*y,(x*y)*z)),
\end{aligned}$$

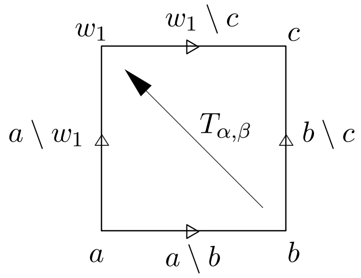

Figure 2: The map $T_{\alpha,\beta}$ assigned to the edges of a quadrilateral

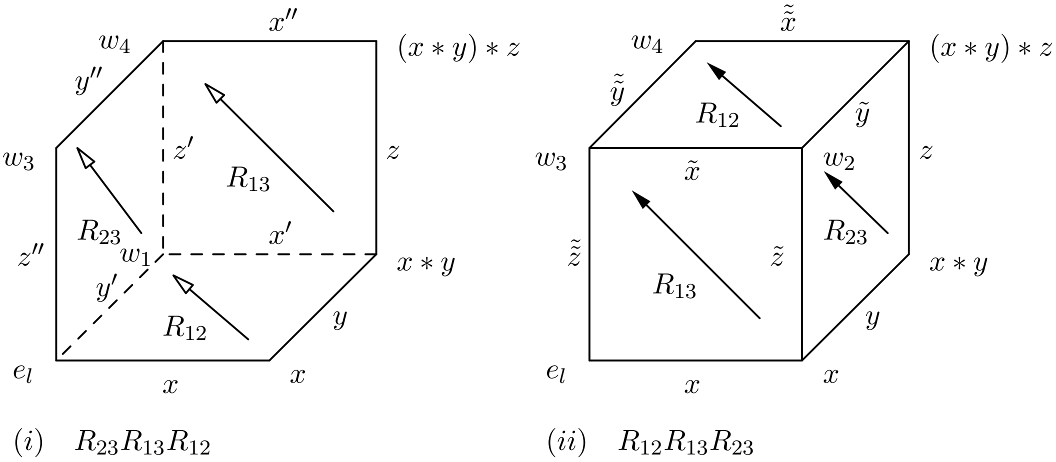

Figure 3: Cubic representation of the Yang–Baxter property

where for w_3 and w_4 the last equality is obtained by equations (7) and (8) respectively.

Using the previous relations, the definition of $T_{\alpha,\beta}$ and the corresponding symmetry condition $\lambda\mu_{\alpha,\beta}(a,b,c) = \mu_{\alpha,\beta}(\lambda a, \lambda b, \lambda c)$ for proposition 3.3 and $\lambda\mu_{\alpha,\beta}(a,b,c) = \mu_{\alpha,\beta}(\frac{a}{\lambda}, \lambda b, \frac{c}{\lambda})$ for proposition 3.4, we can verify that

$$(x', y') = T_{\alpha,\beta}(e_l, x, x*y),\ (x'', z') = T_{\alpha,\gamma}(w_1, x*y, (x*y)*z),\ (y'', z'') = T_{\beta,\gamma}(e_l, w_1, w_4),$$
$$(\tilde{y}, \tilde{z}) = T_{\beta,\gamma}(x, x*y, (x*y)*z),\ (\tilde{x}, \tilde{\tilde{z}}) = T_{\alpha,\gamma}(e_l, x, w_2),\ (\tilde{\tilde{x}}, \tilde{\tilde{y}}) = T_{\alpha,\beta}(w_3, w_2, (x*y)*z),$$

and from (13) we derive $x'' = w_4 \setminus ((x*y)*z) = \tilde{\tilde{x}}$, $y'' = w_3 \setminus w_4 = \tilde{\tilde{y}}$ and $z'' = w_3 = \tilde{\tilde{z}}$. $\square$

A similar construction on the loop $(\mathbb{C}, *)$, with $a*b = b-a$, satisfying an equivalent symmetry condition, can be applied as well. In this case the corresponding symmetry condition of proposition 3.4 becomes

$$\lambda + \mu_{\alpha,\beta}(a,b,c) = \mu_{\alpha,\beta}(a-\lambda, \lambda+b, c-\lambda), \tag{14}$$

and the induced parametric YB map is

$$R_{\alpha,\beta}(x,y) = (\mu_{\alpha,\beta}(0,x,y-x) + y - x, \mu_{\alpha,\beta}(0,x,y-x)). \tag{15}$$

The proof is similar with the proof of proposition 3.4 by considering the operation $a*b = b-a$, the left identity element $e_l = 0$ and the left division $a \setminus b = a+b$.

We summarize the latter results in the next proposition.

Proposition 3.5. *Let $(L, \mu_{\alpha,\beta})$ be a 3D compatible ternary system on an Abelian group $(L, \cdot)$, such that*

$$\lambda \cdot \mu_{\alpha,\beta}(a, b, c) = \mu_{\alpha,\beta}(a \cdot \lambda^{-1}, \lambda \cdot b, c \cdot \lambda^{-1}), \quad \textit{for every } \lambda \in L \textit{ and } \alpha, \beta \in I, \tag{16}$$

then the map

$$R_{\alpha,\beta}(x, y) = (y \cdot x^{-1} \cdot \mu_{\alpha,\beta}(e, x, y \cdot x^{-1}), \mu_{\alpha,\beta}(e, x, y \cdot x^{-1})) \tag{17}$$

is a parametric YB map.

Proof. On any group $(L, \cdot)$ we can define the binary operation $* : L \times L \mapsto L$, $a * b = b \cdot a^{-1}$. It is easy to verify that $(L, *)$ is a qausigroup with left identity element the identity element e of the group $(L, \cdot)$ and left division $a \setminus b = a \cdot b$. The proof follows as in Prop. 3.4 for

$$T_{\alpha,\beta} : (a, b, c) \mapsto (\mu_{\alpha,\beta}(a, b, c) \cdot c, a \cdot \mu_{\alpha,\beta}(a, b, c)) \text{ and } R_{\alpha,\beta}(x, y) = T_{\alpha,\beta}(e, x, x * y),$$

using the symmetry condition (16) and the fact that $(L, \cdot)$ is Abelian. □

We have to notice that the previous propositions are inspired by the symmetry method used in [14] in order to obtain YB maps out of 3D-compatible quad-graph equations. We provide several examples in the next section.

3.1 YB maps from the ABS classification list

We consider 3D compatible ternary systems from two integrable equations on quad-graphs of the ABS classification list [1]. In each case, the corresponding quasigroup structure that we consider is chosen according to the symmetry condition that they satisfy.

3.1.1 YB maps from the Q_1 equation

The equation

$$\alpha(a - w)(b - c) - \beta(a - b)(w - c) = 0, \quad a, b, c, w, \alpha, \beta \in \mathbb{C}$$

(the equation Q_1 of the Q list of Adler, Bobenko, Suris [1]) is 3D consistent. By solving it with respect to w, we define the 3D compatible ternary system on $(\mathbb{C}, +)$

$$w = \mu_{\alpha,\beta}(a, b, c) = \frac{\alpha a(b - c) + \beta c(a - b)}{\alpha(b - c) + \beta(a - b)}, \tag{18}$$

which is homogeneous, i.e. $\mu_{\alpha,\beta}(\lambda + a, \lambda + b, \lambda + c) = \lambda + \mu_{\alpha,\beta}(a, b, c)$. The corresponding YB map (10) is

$$R_{\alpha,\beta}(x, y) = (\frac{\alpha y(x + y)}{\beta x + \alpha y}, \frac{\beta x(x + y)}{\beta x + \alpha y}).$$

Furthermore, equation (18) defines a homogeneous 3D compatible ternary system on $(\mathbb{C}^*, \cdot)$, since $\mu_{\alpha,\beta}(\lambda a, \lambda b, \lambda c) = \lambda \mu_{\alpha,\beta}(a, b, c)$. In this case the corresponding YB map of Prop. 3.3 is

$$\begin{aligned} R_{\alpha,\beta}(x, y) &= (\frac{xy}{\mu_{\alpha,\beta}(1, x, xy)}, \mu_{\alpha,\beta}(1, x, xy)) \\ &= (y\frac{\alpha xy + (\beta - \alpha)x - \beta}{\beta xy + (\alpha - \beta)y - \alpha}, x\frac{\beta xy + (\alpha - \beta)y - \alpha}{\alpha xy + (\beta - \alpha)x - \beta}), \end{aligned}$$

that is the H_{II} YB map of the H list of [12].

3.1.2 YB maps from the dKdV

The discrete Korteweg-de-Vries equation

$$(c-a)(b-w)=\alpha-\beta, \quad a,b,c,w,\alpha,\beta\in\mathbb{C}$$

is a 3D consistent equation. If we solve it with respect to w, we define the ternary operation $\mu_{\alpha,\beta}$ by

$$w=\mu_{\alpha,\beta}(a,b,c)=b-\frac{\alpha-\beta}{c-a},$$

that satisfies the equations (7) and (8). Also, $\mu_{\alpha,\beta}(\lambda+a,\lambda+b,\lambda+c)=\lambda+\mu_{\alpha,\beta}(a,b,c)$, The corresponding YB map of Prop. 3.3 is

$$R_{\alpha,\beta}(x,y)=((x+y-\mu_{\alpha,\beta}(0,x,x+y)),\mu_{\alpha,\beta}(0,x,x+y))=(y+\frac{\alpha-\beta}{x+y},x-\frac{\alpha-\beta}{x+y}),$$

i.e. the Adler's map.

Furthermore, $\lambda\mu_{\alpha,\beta}(a,b,c)=\mu_{\alpha,\beta}(\frac{a}{\lambda},\lambda b,\frac{c}{\lambda})$. By considering the 3D compatible ternary system $(L,\mu_{\alpha,\beta})$ on the quasigroup $(L,*)$ of Prop. 3.4, we derive the parametric YB map

$$R_{\alpha,\beta}(x,y)=(\frac{y}{x}\mu_{\alpha,\beta}(1,x,\frac{y}{x}),\mu_{\alpha,\beta}(1,x,\frac{y}{x}))=(y(1+\frac{\alpha-\beta}{x-y}),x(1+\frac{\alpha-\beta}{x-y})),$$

which is the F_{IV} map of the F list of the classification in [2].

Finally, $\mu_{\alpha,\beta}$ satisfies also the symmetry condition (14). If we consider the 3D compatible ternary system $(\mathbb{C},\mu_{\alpha,\beta})$ on the Loop $(\mathbb{C},*)$, with $a*b=b-a$, then from (15) we derive the YB map

$$R_{\alpha,\beta}(x,y)=(y+\frac{\alpha-\beta}{x-y},x+\frac{\alpha-\beta}{x-y}),$$

that is the F_V map of the F list of the classification in [2].

4 From YB maps to 3D Consistent quad-graph equations

In this section we study the inverse problem, that is to derive 3D compatible ternary systems from YB maps. For dynamical YB maps on quasigroups satisfying an invariance condition, the answer is given by Shibukawa in [17].

Theorem 4.1. (*Shibukawa*) *Let $R_\lambda(x,y)=(\eta_\lambda(y)(x),\xi_\lambda(x)(y))$ be a dynamical YB map on a left quasigroup $(L,\cdot)$, satisfying the invariance condition*

$$(\lambda\xi_\lambda(x)(y))\eta_\lambda(y)(x)=(\lambda x)y, \tag{19}$$

for every $\lambda,x,y\in L$. Then the ternary operation μ on L defined by

$$\mu(a,b,c)=a\xi_a(a\setminus b)(b\setminus c) \tag{20}$$

is 3D-compatible (i.e. satisfies equations (5)).

Since every YB map is a dynamical YB map, we can use this theorem in order to derive 3D compatible ternary systems from YB maps satisfying a corresponding invariance condition. It is easy to verify that all the parametric YB maps that we presented in the previous section satisfy condition (19). In particular, if we denote by $R_{\alpha,\beta}:(x,y)\mapsto(u(x,y),v(x,y))$ each map on the corresponding quasigroup $(L,*)$, then the condition (19) becomes

$$v(x,y)*u(x,y)=x*y. \tag{21}$$

4.1 Multiparametric YB maps and consistency

Next, as application of the last construction, we present first a four parametric YB map obtained by reduction through a compatible constraint of a higher dimensional map introduced in [6] and then apply theorem 4.1 to obtain a four parametric 3D compatible quad-graph equation. We will use the next lemma.

Lemma 4.1. *Let $R_{\alpha,\beta}: X^2 \times X^2 \to X^2 \times X^2$ be a parametric YB map with*

$$R_{\alpha,\beta}(x_1,x_2,y_1,y_2) = (u_1(x_1,x_2,y_1,y_2), u_2(x_1,x_2,y_1,y_2), v_1(x_1,x_2,y_1,y_2), v_2(x_1,x_2,y_1,y_2))$$

(here the functions u_i, v_i, for $i=1,2$, depend on the parameters α, β) and $f_\alpha : X \to X$ a parameter depending function, such that

$$u_2(x_1, f_\alpha(x_1), y_1, f_\beta(y_1)) = f_\alpha(u_1(x_1, f_\alpha(x_1), y_1, f_\beta(y_1))), \tag{22}$$
$$v_2(x_1, f_\alpha(x_1), y_1, f_\beta(y_1)) = f_\beta(v_1(x_1, f_\alpha(x_1), y_1, f_\beta(y_1))), \tag{23}$$

then the map $\tilde{R}_{\alpha,\beta}: X \times X \to X \times X$, defined by

$$\tilde{R}_{\alpha,\beta}(x,y) = (u_1(x, f_\alpha(x), y, f_\beta(y)), v_1(x, f_\alpha(x), y, f_\beta(y)))$$

is a parametric YB map. Furthermore, if $L(x_1,x_2,\alpha)$ is a Lax matrix of $R_{\alpha,\beta}$, then $\tilde{L}(x,\alpha) = L(x, f_\alpha(x), \alpha)$ is a Lax matrix of $\tilde{R}_{\alpha,\beta}$.

Direct computations prove this lemma (appendix). Lemma 4.1 can be generalized on higher dimensional YB maps $R_{\alpha,\beta}: X^n \times X^n \to X^n \times X^n$,

$$R_{\alpha,\beta} : (x_1,...,x_n,y_1,...,y_n) \mapsto (u_1,...,u_n,v_1,...,v_n),$$

with a compatible parametric function $f_\alpha : X^{n-1} \to X$, such that if we replace x_k, y_k by $f_\alpha(x_1,...,x_{k-1},x_{k+1},...x_n)$, $f_\beta(y_1,...,y_{k-1},y_{k+1},...y_n)$ for $k=1,...n$, respectively, then $u_k \mapsto f_\alpha(u_1,...,u_{k-1},u_{k+1},...,u_n)$ and $v_k \mapsto f_\beta(v_1,...,v_{k-1},v_{k+1},...,v_n)$. In this way the new u_i, v_i, $i=1,...,n$, $i \neq k$, give rise to a YB map on $X^{n-1} \times X^{n-1}$.

Now, we consider the YB map

$$\mathcal{R}_{\alpha,\beta} : ((x_1,x_2),(y_1,y_2)) \mapsto ((U_{11},U_{12}),(V_{11},V_{12})),$$

where U_{ij}, V_{ij} denote the corresponding ij elements of the matrices

$$U = (\bar{L}(y_1,y_2;\beta)\bar{L}(x_1,x_2;\alpha) + \frac{1}{\alpha_1\alpha_2}K_\alpha K_\beta)(\bar{L}(y_1,y_2;\beta)K_\alpha + K_\beta\bar{L}(x_1,x_2;\alpha))^{-1}K_\alpha,$$
$$V = K_\alpha^{-1}(\bar{L}(y_1,y_2;\beta)K_\alpha + K_\beta\bar{L}(x_1,x_2;\alpha) - UK_\beta), \quad \text{for } \alpha=(\alpha_1,\alpha_2),\ \beta=(\beta_1,\beta_2),$$
$$K_\alpha = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \quad \text{and} \quad \bar{L}(x_1,x_2;\alpha) = \begin{pmatrix} x_1 & x_2 \\ \frac{\alpha_1-\alpha_2 x_1{}^2}{\alpha_1 x_2} & -\frac{\alpha_2 x_1}{\alpha_1} \end{pmatrix}.$$

This map is "*Case I*" quadrirational symplectic YB map of the classification of binomial Lax matrices presented in [6], for $a_3=-1$ and $a_4=0$, with strong Lax matrix $\mathcal{L}(x_1,x_2,\alpha) = \bar{L}(x_1,x_2;\alpha) - \zeta K_\alpha$ and Poisson bracket

$$\{x_1,x_2\} = -\alpha_1 x_2,\ \{y_1,y_2\} = -\beta_1 y_2,\ \{x_i,y_j\} = 0 \text{ for } i=1,2,\ (\alpha_1,\beta_1,x_2,y_2 \neq 0).$$

By setting $x_1 = y_1 = 0$ we derive $U_{11} = V_{11} = 0$, and the resulting from lemma 4.1 map

$$R_{\alpha,\beta}(x,y) = (y\frac{\beta_1 x + \alpha_2 y}{\alpha_1 x + \beta_2 y}, x\frac{\beta_1 x + \alpha_2 y}{\alpha_1 x + \beta_2 y}), \tag{24}$$

is a parametric YB map with strong Lax matrix

$$L(x;\alpha) = \mathcal{L}(0,x;\alpha) = \begin{pmatrix} -\alpha_1\zeta & x \\ \frac{1}{x} & -\alpha_2\zeta \end{pmatrix}. \tag{25}$$

Furthermore, this YB map satisfies the invariance condition

$$\frac{u(x,y)}{v(x,y)} = \frac{y}{x}, \text{ for } u(x,y) = y\frac{\beta_1 x + \alpha_2 y}{\alpha_1 x + \beta_2 y} \text{ and } v(x,y) = x\frac{\beta_1 x + \alpha_2 y}{\alpha_1 x + \beta_2 y}. \tag{26}$$

If we consider the YB map (24) as a birational map on the quasigroup $L = \mathbb{C} \setminus \{0\}$ with the binary operation $a * b = \frac{b}{a}$, then the invariance condition (26) corresponds to (19) and by theorem 4.1 we obtain the corresponding 3D compatible ternary system on $(L, *)$ with

$$\mu_{\alpha,\beta}(a,b,c) = b\frac{\beta_1 a + \alpha_2 c}{\alpha_1 a + \beta_2 c}, \quad \alpha = (\alpha_1, \alpha_2), \ \beta = (\beta_1, \beta_2), \tag{27}$$

or equivalently, by setting $w = \mu_{\alpha,\beta}(a,b,c)$,

$$w(\alpha_1 a + \beta_2 c) - b(\beta_1 a + \alpha_2 c) = 0. \tag{28}$$

From the other hand, this 3D compatible ternary system on $(L, *)$ satisfies the symmetry condition (11), and the corresponding YB map of Prop. 3.4

$$R_{\alpha,\beta}(x,y) = (\frac{y}{x}\mu_{\alpha,\beta}(1,x,\frac{y}{x}), \mu_{\alpha,\beta}(1,x,\frac{y}{x})),$$

coincides with the YB map (24).

We can reduce the number of parameters of the YB map (24) by setting $\alpha_1 = \beta_1 = c$ or $\alpha_2 = \beta_2 = c$. with c constant. Another interesting reduction of parameters is derived by setting $\alpha_1 = \alpha - r, \ \alpha_2 = \alpha + r, \ \beta_1 = \beta - r, \ \beta_2 = \beta + r$, with r constant. Then the YB map (24) is transformed to

$$\bar{R}_{\alpha,\beta}(x,y) = (y\frac{(\beta - r)x + (\alpha + r)y}{(\alpha - r)x + (\beta + r)y}, x\frac{(\beta - r)x + (\alpha + r)y}{(\alpha - r)x + (\beta + r)y}), \tag{29}$$

with strong Lax matrix $L(x, \alpha - r, \alpha + r)$. Here r is not a YB parameter but just a free parameter. The corresponding 3D compatible ternary system of theorem 4.1,

$$w((\alpha - r)a + (\beta + r)c) - b((\beta - r)a + (\alpha + r)c) = 0,$$

has been introduced in [10] and is a "homotopy" of discrete MKdV and Toda equations.

The YB map (24) (as well as any map derived by reducing the parameters described above) is quadrirational and belongs to the subclass [2 : 2] of [2], nevertheless it is not an involution since $R_{\alpha,\beta} \circ R_{\alpha,\beta} \neq id$. Therefore, it is not equivalent with any map from the F and H families in [12]. We notice that the induced 3D compatible system (27) is not D_4-symmetric. However, it is homogeneous with respect to multiplication and generates another one four parametric YB map (involution this time) according to proposition 3.3,

$$R_{\alpha,\beta}(x,y) = (\frac{xy}{\mu_{\alpha,\beta}(1,x,xy)}, \mu_{\alpha,\beta}(1,x,xy)) = (y\frac{\alpha_1 + \beta_2 xy}{\beta_1 + \alpha_2 xy}, x\frac{\beta_1 + \alpha_2 xy}{\alpha_1 + \beta_2 xy}),$$

that satisfies the invariant condition (21).

4.2 3D Consistency on $GL_2(\mathbb{C})$

We conclude this section by applying theorem 4.1 on the general YB map presented in [6]. Let $X,\ Y$ be two generic elements of $GL_2(\mathbb{C})$ and $K:\mathbb{C}^d \to GL_2(\mathbb{C})$ a d-parametric family of commuting matrices. It is enough to consider as K one of the two cases of the classification in [6] (three cases in $GL_2(\mathbb{R})$ respectively). Then the map

$$\mathcal{R}_{\alpha,\beta}(X,Y) = (U_{\alpha,\beta}(X,Y), V_{\alpha,\beta}(X,Y)), \quad \text{where}$$

$$U_{\alpha,\beta}(X,Y) = (f_2^\alpha(X)YX - f_0^\alpha(X)K_\alpha K_\beta)(f_2^\alpha(X)(YK_\alpha + K_\beta X) - f_1^\alpha(X)K_\alpha K_\beta)^{-1}K_\alpha,$$
$$V_{\alpha,\beta}(X,Y) = K_\alpha^{-1}(YK_\alpha + K_\beta X - U_{\alpha,\beta}(X,Y)K_\beta)$$

and f_i^α, $i=0,1,2$ are defined by the coefficients of the characteristic polynomial

$$\det(X - \zeta K_\alpha) = f_2^\alpha(X)\zeta^2 - f_1^\alpha(X)\zeta + f_0^\alpha(X),$$

is a quadrirational parametric YB map that satisfies the invariant conditions

$$(U_{\alpha,\beta}(X,Y) - \zeta K_\alpha)(V_{\alpha,\beta}(X,Y) - \zeta K_\beta) = (Y - \zeta K_\beta)(X - \zeta K_\alpha),$$
$$f_i^\alpha(U_{\alpha,\beta}(X,Y)) = f_i^\alpha(X),\ \ f_i^\beta(V_{\alpha,\beta}(X,Y)) = f_i^\beta(Y),$$

for $i = 0,1,2$.

Proposition 4.2. *The ternary operation*

$$\mu_{\alpha,\beta}(A,B,C) = V_{\alpha,\beta}(BA^{-1}, CB^{-1})A \tag{30}$$

defines a 3D compatible ternary system on $GL_2(\mathbb{C})$.

Proof. On $GL_2(\mathbb{C})$ we define the binary operation $A * B = BA$. So, $A \setminus B = BA^{-1}$. Now from the first invariant condition of $\mathcal{R}_{\alpha,\beta}$ we have that

$$V_{\alpha,\beta}(X,Y) * U_{\alpha,\beta}(X,Y) = U_{\alpha,\beta}(X,Y)V_{\alpha,\beta}(X,Y) = YX = X * Y,$$

which is equivalent with (19). So, by theorem 4.1, we conclude that $(GL_2(\mathbb{C}), \mu_{\alpha,\beta})$ with

$$\mu_{\alpha,\beta}(A,B,C) = A * V_{\alpha,\beta}(A \setminus B, B \setminus C) = V_{\alpha,\beta}(BA^{-1}, CB^{-1})A$$

is a 3D compatible ternary system. □

This 3D compatible ternary system can be generalized in $GL_n(\mathbb{C})$ by considering the corresponding YB maps on $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ from the recursive formulae presented in [6].

5 Conclusions

We showed that the conditions of the construction of dynamical Yang-Baxter maps out of ternary systems introduced by Shibukawa are equivalent with the 3D consistency property of equations on quadrilaterals. Moreover, certain symmetry conditions on the 3D compatible ternary systems drop the dynamical parameter and yield (plain) YB maps. It is clear that the underlying quasigroup structure of the evolution space is crucial in this construction and must be compatible with the symmetry condition of the initial ternary system. From the

other hand, parametric YB maps satisfying an invariance condition give rise to 3D compatible systems. In this case the suitable quasigroup structure can be traced from the invariance condition of the map. Nevertheless, other quasigroup laws with additional symmetry conditions on 3D consistent equations will lead to new parametric YB maps with invariance conditions and vice versa.

Incidentally, we saw in section 4.1 that YB maps can be derived by considering some compatible constraints (Lemma 4.1) on higher dimensional Poisson YB maps. The question that arises is how to find these constraints and also how to find new compatible Poisson structures in order to study the integrability of the corresponding transfer maps on lattices (see for example [11, 19, 7]).

Finally, the relation of the presented non-involution YB maps with the maps classified in [12], as well as the question of existence and significance of multiparameter extensions of known YB maps and 3D compatible quad-graph equations, is an interesting issue that deserves further investigation.

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Appendix

The proof of Lemma 4.1.

Proof. Suppose that $\tilde{R}^{23}_{\beta,\gamma}\tilde{R}^{13}_{\alpha,\gamma}\tilde{R}^{12}_{\alpha,\beta}(x,y,z)=(x',y',z')$, $\tilde{R}^{12}_{\alpha,\beta}\tilde{R}^{13}_{\alpha,\gamma}\tilde{R}^{23}_{\beta,\gamma}(x,y,z)=(x'',y'',z'')$ and

$$R^{23}_{\beta,\gamma}R^{13}_{\alpha,\gamma}R^{12}_{\alpha,\beta}(x_1,x_2,y_1,y_2,z_1,z_2)=(\bar{x}_1,\bar{x}_2,\bar{y}_1,\bar{y}_2,\bar{z}_1,\bar{z}_2),$$
$$R^{12}_{\alpha,\beta}R^{13}_{\alpha,\gamma}R^{23}_{\beta,\gamma}(x_1,x_2,y_1,y_2,z_1,z_2)=(\bar{\bar{x}}_1,\bar{\bar{x}}_2,\bar{\bar{y}}_1,\bar{\bar{y}}_2,\bar{\bar{z}}_1,\bar{\bar{z}}_2).$$

Then $x'=u_1(u_1(x,f_\alpha(x),y,f_\beta(y)),f_\alpha(u_1(x,f_\alpha(x),y,f_\beta(y))),z,f_\gamma(z))$, while

$$\begin{aligned}x''=u_1(&u_1(x,f_\alpha(x),v_1(y,f_\beta(y),z,f_\gamma(z)),f_\gamma(v_1(y,f_\beta(y),z,f_\gamma(z)))),\\&f_\alpha(u_1(x,f_\alpha(x),v_1(y,f_\beta(y),z,f_\gamma(z)),f_\gamma(v_1(y,f_\beta(y),z,f_\gamma(z))))),\\&u_1(y,f_\beta(y),z,f_\gamma(z)),f_\beta(u_1(y,f_\beta(y),z,f_\gamma(z)))).\end{aligned}$$

If we set $x=x_1,\ y=y_1,\ z=z_1,\ f_\alpha(x_1)=x_2,\ f_\beta(y_1)=y_2,\ f_\gamma(z_1)=z_2$, then from (22),(23) we have that
$f_\alpha(u_1(x,f_\alpha(x),y,f_\beta(y)))=u_2(x_1,x_2,y_1,y_2)$, $f_\beta(u_1(y,f_\beta(y),z,f_\gamma(z))))=u_2(y_1,y_2,z_1,z_2)$,
$f_\gamma(v_1(y,f_\beta(y),z,f_\gamma(z))))=v_2(y_1,y_2,z_1,z_2)$,
$f_\alpha(u_1(x,f_\alpha(x),v_1(y,f_\beta(y),z,f_\gamma(z)),f_\gamma(v_1(y,f_\beta(y),z,f_\gamma(z)))))=$
$u_2(x_1,x_2,v_1(y_1,y_2,z_1,z_2),v_2(y_1,y_2,z_1,z_2))$, so $x'_1=\bar{x}_1$, $x''_1=\bar{\bar{x}}_1$ and since $\bar{x}_1=\bar{\bar{x}}_1$, we derive that $x'_1=x''_1$ or $x'=x''$. In a similar way we can show that $y'=y''$ and $z'=z''$.

Furthermore, since

$$L(u_1(x_1,x_2,y_1,y_2),u_2(x_1,x_2,y_1,y_2),\alpha)L(v_1(x_1,x_2,y_1,y_2),v_2(x_1,x_2,y_1,y_2),\beta) = \\ L(x_1,x_2,\alpha)L(y_1,y_2,\beta), \text{ for every } x_i,y_i \in X, \ i=1,2,$$

then
$L(u_1(x,f_\alpha(x),y,f_\beta(y)),u_2(x,f_\alpha(x),y,f_\beta(y)),\alpha)L(v_1(x,f_\alpha(x),y,f_\beta(y)),v_2(x,f_\alpha(x),y,f_\beta(y)),\beta)$
$= L(x,f_\alpha(x),\alpha)L(y,f_\beta(y),\beta)$, for every $x,y\in X$, and from (22),(23) we get that

$$L(u_1(x,f_\alpha(x),y,f_\beta(y)),f_\alpha(u_1(x,f_\alpha(x),y,f_\beta(y))))\cdot \\ L(v_1(x,f_\alpha(x),y,f_\beta(y)),f_\beta(v_1(x,f_\alpha(x),y,f_\beta(y)))) = L(x,f_\alpha(x),\alpha)L(y,f_\beta(y),\beta),$$

which means that $\tilde{L}(x,\alpha)=L(x,f_\alpha(x),\alpha)$ is a Lax matrix of the YB map $\tilde{R}_{\alpha,\beta}$.

□

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